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NORM ACHIEVED TOEPLITZ AND HANKEL OPERATORS

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Let μ be the normalized Lebesgue measure on the Borel sets of the unit circle in the complex plane $\mathbb{C}$. For a $\varphi \in L^\infty$ the Laurent operator L_φ is given by $L_\varphi f = \varphi f$ for $f \in L^2$ as the multiplication operator on L^2. And the Laurent operator induces, in a natural way, twin operators on H^2 called the Toeplitz operator T_φ given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 and the Hankel operator H_φ given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$ where J is the unitary operator on L^2 defined by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \cdots$.

The following results are known.

Proposition 1. If φ is a non-constant function in L^∞, then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_\varphi{}^*)} = \emptyset$ where $\sigma_p(T_\varphi)$ denotes the point spectrum of T_φ and the bar denotes the complex conjugate.

Proposition 2. If φ and ψ are in H^∞, then $T_\varphi H^2 \subseteq T_\psi H^2$ if and only if there exists a $g \in H^\infty$ uniquely, up to a unimodular constant, such that $T_\varphi = T_\psi T_g = T_{\psi g}$. And then $\varphi = \psi g$. Particularly, if φ and ψ are inner, then g is also inner.

Proposition 3. H_φ has the following properties.

(1) $T_z{}^* H_\varphi = H_\varphi T_z$

(2) $H_\varphi{}^* = H_{\varphi^*}$ where $\varphi^*(z) = \overline{\varphi(\overline{z})}$

(3) $H_{\alpha\varphi+\beta\psi} = \alpha H_\varphi + \beta H_\psi, \quad \alpha,\ \beta \in \mathbb{C}$

(4) $H_\varphi = O$ if and only if $(I-P)\varphi = o$ (i.e., $\varphi \in H^\infty$)

(5) $\|H_\varphi\| = \min\{\|\varphi + \psi\|_\infty : \psi \in H^\infty\}$

Proposition 4. $H_\psi{}^* H_\varphi = T_{\overline{\psi}\varphi} - T_{\overline{\psi}} T_\varphi$.

Proposition 5. For any $\psi \in H^\infty$, $H_\varphi T_\psi = H_{\varphi\psi}$.

Lemma 1. The following assertions are equivalent.

(1) $\mathcal{N}_{H_\varphi} \neq \{o\}$.

(2) $[H_\varphi H^2]^{\sim L^2} \neq H^2$.

(3) $\varphi = \bar{g}h$ for some inner function g and $h \in H^\infty$ such that g and h have no common non-constant inner factor.

Proof. $\underline{(1) \rightleftharpoons (2)}$;

$$\begin{aligned} H_\varphi f = o \quad &\rightleftharpoons \quad \varphi f \in H^2 \quad \rightleftharpoons \quad \varphi^* f^* \in H^2 \\ \rightleftharpoons \quad H_\varphi^* f^* &= H_{\varphi^*} f^* = o \quad \rightleftharpoons \quad f^* \perp [H_\varphi H^2]^{\sim L^2}. \end{aligned}$$

$\underline{(1) \to (3)}$; Since $\mathcal{N}_{H_\varphi}$ is a non-zero invariant subspace of T_z by Proposition 3, $\mathcal{N}_{H_\varphi} = T_g H^2$ for some inner function g. Hence, by Proposition 5, $O = H_\varphi T_g = H_{\varphi g}$ and $\varphi g = h \in H^\infty$ by Proposition 3(4). Therefore $\varphi = \bar{g}h$. If $g = g_1 g_2$ and $h = g_1 h_1$ for some non-constant inner function g_1 and g_2, $h_1 \in H^\infty$, then, by Propositions 2 and 5,

$$T_{g_2} H^2 \supset T_g H^2 = \mathcal{N}_{H_\varphi} = \mathcal{N}_{H_{\overline{g_2} h_1}} \supseteq T_{g_2} H^2$$

and this is a contradiction. Therefore g and h have no common non-constant inner factor.

$\underline{(3) \to (1)}$; By Propositions 5 and 3(4), we have $H_\varphi T_g H^2 = H_{\varphi g} H^2 = H_h H^2 = \{o\}$ and $\mathcal{N}_{H_\varphi} \supseteq T_g H^2 \neq \{o\}$. □

Theorem 1. The Toeplitz operator T_φ is norm-achieved (i.e., $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} \neq \{o\}$) if and only if $\frac{\varphi}{\|T_\varphi\|} = g$ for some $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(H_g)$.

And, in this case, $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} = \mathcal{N}_{H_g}$ and it is invariant under T_z by Proposition 3(1).

Proof. $\underline{(\to)}$; If $\|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2$ for some non-zero $f \in H^2$, then we have, for $g = \frac{\varphi}{\|T_\varphi\|}$,

$$\|f\|_2 = \|T_{\frac{\varphi}{\|T_\varphi\|}} f\|_2 = \|T_g f\|_2 = \|P L_g f\|_2 \leq \|L_g f\|_2 \leq \|f\|_2$$

because $\|L_g\| = \|T_g\| = \frac{\|T_\varphi\|}{\|T_\varphi\|} = 1$. Hence $T_g{}^*T_gf = f$ and $PL_gf = L_gf$ and hence $H_gf = J(I-P)L_gf = o$ (i.e., $0 \in \sigma_p(H_g)$). Since, by Proposition 4, $H_g{}^*H_g = T_{|g|^2} - T_{\bar{g}}T_g$, we have $T_{|g|^2}f = f$ (i.e., $1 \in \sigma_p(T_{|g|^2})$) and, by Proposition 1, $|g|^2$ is constant and $|g| = 1$ a.e.

$(\leftarrow)$; Since $\|T_g\| = \frac{\|T_\varphi\|}{\|T_\varphi\|} = 1$ and since, by Proposition 4, $H_g{}^*H_g = I - T_{\bar{g}}T_g$, we have $T_g{}^*T_gf = f$ for all $f \in \mathcal{N}_{H_g}$ and hence $\|T_gf\|_2 = \|f\|_2$. Therefore $\|T_\varphi f\|_2 = \|T_\varphi\|\|T_gf\|_2 = \|T_\varphi\|\|f\|_2$.

The last assertion is clear. In fact, $(\rightarrow)$ implies that

$$\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\|\|f\|_2\} \subseteq \mathcal{N}_{H_g}$$

and $(\leftarrow)$ implies the converse inclusion. □

Corollary 1. T_φ is norm-achieved if and only if $\frac{\varphi}{\|T_\varphi\|} = \bar{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor.

And, in this case, $\emptyset \neq \sigma(T_\varphi) \cap \{\lambda \in \mathbb{C} : \|T_\varphi\| = |\lambda|\} \subseteq \sigma_c(T_\varphi)$ where $\sigma_c(T_\varphi)$ denotes the continuous spectrum of T_φ.

Proof. By Theorem 1, T_φ is norm-achieved if and only if $\frac{\varphi}{\|T_\varphi\|} = g$ for some $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(H_g)$. And then, by Lemma 1, $\mathcal{N}_{H_g} \neq \{o\}$ if and only if $g = \bar{q}h$ for some inner function q and $h \in H^\infty$ such that q and h have no common non-constant inner factor. Since $|g| = 1$ a.e. if and only if $|h| = 1$ a.e. and h is also an inner function.

It is known that $\sigma(L_\varphi) \subseteq \sigma(T_\varphi)$ and since L_g is unitary because $|g| = 1$ a.e., we have $\sigma(T_\varphi) \cap \{\lambda \in \mathbb{C} : \|T_\varphi\| = |\lambda|\} \neq \emptyset$. If $T_gx = e^{i\theta}x$ for some $\theta \in [0, 2\pi)$ and non-zero $x \in H^2$, then

$$\|x\| = \|T_gx\| = \|T_q{}^*T_hx\| \leq \|T_hx\| = \|x\|$$

and $e^{i\theta}T_qx = T_qT_gx = T_qT_q{}^*T_hx = T_hx$. Since $T_h - e^{i\theta}T_q$ is hyponormal, $(T_h - e^{i\theta}T_q)x = o$ implies $(T_h - e^{i\theta}T_q)^*x = o$ and this contradicts Proposition 1 and hence $\sigma(T_\varphi) \cap \{\lambda \in \mathbb{C} : \|T_\varphi\| = |\lambda|\} \subseteq \sigma_c(T_\varphi)$ because

$$\sigma_r(T_\varphi) \cap \{\lambda \in \mathbb{C} : \|T_\varphi\| = |\lambda|\} = \emptyset$$

where $\sigma_r(T_\varphi)$ denotes the residual spectrum of T_φ. □

In the case of Hankel operators, we have the following.

Theorem 2. The Hankel operator H_φ is norm-achieved (i.e., $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} \neq \{o\}$) if and only if $\frac{\varphi}{\|H_\varphi\|} = g + \psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(T_g)$.

And, in this case, $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} = \mathcal{N}_{T_g}$.

Proof. $\underline{(\rightarrow)}$; By Proposition 3, there exists a $g \in L^\infty$ such that $H_{\frac{\varphi}{\|H_\varphi\|}} = H_g$ and $\|H_g\| = \|g\|_\infty$. And then $H_{\frac{\varphi}{\|H_\varphi\|} - g} = O$ and $\psi = \frac{\varphi}{\|H_\varphi\|} - g \in H^\infty$ by Proposition 3. If $\|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2$ for some non-zero $f \in H^2$, then we have

$$\|f\|_2 = \|H_{\frac{\varphi}{\|H_\varphi\|}} f\|_2 = \|H_g f\|_2 = \|(I-P)L_g f\|_2 \leq \|L_g f\|_2 \leq \|f\|_2$$

because $\|L_g\| = \|g\|_\infty = \|H_g\| = \|H_{\frac{\varphi}{\|H_\varphi\|}}\| = \frac{\|H_\varphi\|}{\|H_\varphi\|} = 1$. Hence $H_g{}^* H_g f = f$ and $(I-P)L_g f = L_g f$ and hence $T_g f = PL_g f = o$ (i.e., $0 \in \sigma_p(T_g)$). Since, by Proposition 4, $H_g{}^* H_g = T_{|g|^2} - T_{\overline{g}} T_g$, we have $T_{|g|^2} f = f$ (i.e., $1 \in \sigma_p(T_{|g|^2})$) and, by Proposition 1, $|g|^2$ is constant and $|g| = 1$ a.e.

$\underline{(\leftarrow)}$; By Proposition 3, $\|H_g\| = \|H_{\frac{\varphi}{\|H_\varphi\|}}\| = \frac{\|H_\varphi\|}{\|H_\varphi\|} = 1$. Since, by Proposition 4, $H_g{}^* H_g = I - T_{\overline{g}} T_g$, we have $H_g{}^* H_g f = f$ for all $f \in \mathcal{N}_{T_g}$ and hence $\|H_g f\|_2 = \|f\|_2$. Therefore, by Proposition 3,

$$\|H_\varphi f\|_2 = \|H_{\|H_\varphi\| g} f\|_2 = \|H_\varphi\|\|H_g f\|_2 = \|H_\varphi\|\|f\|_2.$$

The last assertion of the theorem is clear. In fact, $(\rightarrow)$ implies that

$$\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} \subseteq \mathcal{N}_{T_g}$$

and $(\leftarrow)$ implies the converse inclusion. $\square$